O'Neill Section 2.2

Exercise 5 - Suppose that β_1 and β_2 are unit-speed reparametrizations of the same curve α. Show that there is a number s_0 such that $\beta_2(s) = \beta_1(s + s_0)$ for all s. What is the geometric significance of s_0?

Proof. If β is a reparametrization of α, then $\beta'(s(t)) = c(t)\alpha'(t)$, $c : I \to \mathbb{R}^+$. If β is unit speed, then $||\beta|| = 1$. Fix $t \in I$. Apply the above facts to β_1 and β_2 to conclude (arguing with magnitude of the vectors) that:

$$\beta_1'(s_1(t)) = c_t\alpha'(t) = \beta_2'(s_2(t)).$$

Then since t was arbitrary, we get that $\beta_1'(s_1(t)) = \beta_2'(s_2(t))$.
Also note that: $\alpha'(t) = s_i'(t)\beta_i'(s_i(t))$ for $i = 1, 2$. Since $\beta_1'(s_1(t)) = \beta_2'(s_2(t))$ we have:

$$s_1'(t) = s_2'(t).$$

This gives that $s_1(t) + k = s_2(t)$. Let $k = s_0$. Defining $s_1 := s$ we get that $s_2 = s + s_0$.
$\therefore \beta_1(s) = \beta_2(s + s_0)$ $\forall s$ as desired.
The geometric significance of s_0 is that is just starts the curve at a different point along the curve. $\square$